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AN APPLICATION OF A GENERALIZED NéMETH FIXED POINT THEOREM IN HADAMARD MANIFOLDS

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Abstract. In this paper, as an application of a multivalued generalization of the Németh fixed point theorem, we will prove a new existence theorem of Nash equilibrium for a generalized game $\mathcal{G} = (X_i; A_i, P_i)_{i\in I}$ with geodesic convex values in Hadamard manifolds.

1. Introduction

In the last decades, several important concepts of nonlinear analysis have been extended from Euclidean spaces to Riemannian manifold settings in order to go further in the studies of convex analysis, fixed point theory, variational problems, and related topics. The motivation of such studies comes from nonlinear phenomena which require the presence of a non-convex or non-linear structure for the ambient space; e.g., see Kim [4-7], Kristály [8-10], Li et al. [11], Németh [12], Udrişte [14], and references therein.

In 2003, using the Brouwer fixed point theorem as a proving tool, Németh [12] first proved a basic fixed point theorem for continuous maps on a compact geodesic convex subset of a Hadamard manifold, and he proved the existence of solutions for variational inequalities in a Hadamard manifold. Since then, using the Németh fixed point theorem, several authors investigate various applications of variational inequalities, minimax inequalities, and equilibrium problems in Hadamard manifolds, e.g., see [2,8-11].

In a recent paper [4], the author proved a multivalued generalization of the Németh fixed point theorem by replacing the Brouwer fixed

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point theorem for single-valued continuous functions suitably by the Begle fixed point theorem for upper semicontinuous multimaps. And, the author shows that this result can be a useful multivalued tool for proving the existence theorem of multivalued nonlinear problems in Hadamard manifolds as in [6].

In this paper, as an application of a multivalued generalization of the Németh fixed point theorem, we will prove a new existence theorem of Nash equilibrium for a generalized game $\mathcal{G} = (X_i; A_i, P_i)_{i \in I}$ with geodesic convex values in Hadamard manifolds. Also, we give an example of 2-person game which is suitable for our theorem but the previous equilibrium existence theorems can not be applied.

2. Preliminaries

We begin with some basic definitions and terminologies on Riemannian manifolds in [2,9,11,14]. Let M be a complete finite dimensional Riemannian manifold with the Levi-Civita connection ∇ on M. Let $x \in M$ and let T_xM denote the tangent space at x to M. For $x,\ y \in M$, let $\gamma_{x,y} : [0,1] \to M$ be a piecewise smooth curve joining x to y. Then, a curve $\gamma_{x,y}$ (γ for short) is called a *geodesic* if $\gamma(0) = x$, $\gamma(1) = y$, and $\nabla_{\dot\gamma}\dot\gamma = 0$ for all $t \in [0,1]$ (here $\dot\gamma$ denotes $d\gamma(t)/dt$). A geodesic $\gamma_{x,y} : [0,1] \to M$ joining x to y is *minimal* if its arc-length equals its Riemannian distance between x and y. And, M is called a *Hadamard manifold* if M is a simply connected complete Riemannian manifold of non-positive sectional curvature. In a Hadamard manifold, the geodesic between any two points is unique, and the exponential map at each point of M is a global diffeomorphism. Therefore all convexities in a Hadamard manifold as in [10] coincide.

Let $I = \{1, \cdots, n\}$ be a finite index set. For each $i \in I$, if (M_i, g_i) be a finite dimensional Hadamard manifold, then the standard arguments shows that the product manifold $(\mathbf{M}, \mathbf{g})$ of Hadamard manifolds (M_i, g_i) equipped with product geodesic and exponential map is also a finite dimensional Hadamard manifold (e.g., see [6,9]). For each $i \in I$, let X_i be a nonempty geodesic convex subset of a finite dimensional Hadamard manifold M, and $X := \Pi_{i \in I} X_i$ be a subset of a Hadamard manifold $\mathbf{M}$ which is a product space of M equipped with standard geodesic and exponential map. Consequently, X is a geodesic convex set in the product manifold $\mathbf{M} = \Pi_{i\in I} M$ endowed with its natural (warped-) product metric (with the constant weight functions), e.g., see Kristály [9, p.674].

Let X be a nonempty subset of a Riemannian manifold M, we shall denote by 2^X the family of all subsets of X. If $T : X \to 2^M$ and $S : X \to 2^M$ are multimaps (or correspondences), then $S \cap T : X \to 2^M$ is a correspondence defined by $(S \cap T)(x) = S(x) \cap T(x)$ for each $x \in X$. When a multimap $T : X \to 2^X$ is given, we shall denote $T^{-1}(y) := \{x \in X \mid y \in T(x)\}$ for each $y \in X$. A multimap T has *open graph* in X if the graph $Gr\,T := \{(x, y) \in X \times X \mid x \in X \text{ and } y \in T(x)\}$ is open in $X \times X$. When a multimap $T_i : X \to 2^{X_i}$ has open graph in $X = \Pi_{i\in I} X_i$ for each $i \in I$, and let $T : X \to 2^X$ be a multimap defined by $T(x) := \Pi_{i\in I} T_i(x)$ for each $x \in X$, then it is easy to see that the graph of T is open in $X \times X$.

Recall the following concept which generalize the convex condition in linear spaces to Riemannian manifolds:

Definition 2.1. A nonempty subset X of a Riemannian manifold M is said to be *geodesic convex* if for any x, $y \in X$, the geodesic joining x to y is contained in X. For an arbitrary subset C of M, the minimal geodesic convex subset which contains C is called the *geodesic convex hull* of C, and denoted by $Gco(C)$.

Then the above definition of geodesic convex hull in a Riemannian manifold M overcomes the delicate problems of geodesic convexity remarked in [10]. As shown in [2], note that $Gco\,(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$, and $C_n = \{z \in \gamma_{x,y} \mid x, y \in C_{n-1}\}$ for each $n \in \mathbb{N}$.

If S is geodesic convex, then $Gco\,(S) = S$, and the intersection of two geodesic convex subsets of M is clearly geodesic convex; but the union of two geodesic convex subsets need not be geodesic convex.

Here, we note that the following operations are essential in proving the geodesic convexity:

Lemma 2.2. [7] Let X and Y be nonempty subsets of a Hadamard manifold M, $X \cap Y$ be nonempty, and $Gco\,(X)$ and $Gco\,(Y)$ be two geodesic convex hull of X and Y in M, respectively. Then we have

(1) $Gco\,(X) \cap Gco\,(Y)$ is geodesic convex;
(2) $Gco\,(X \cap Y)$ is a geodesic convex subset of $Gco\,(X) \cap Gco\,(Y)$;
(3) $Gco\,(X \times Y)$ is a geodesic convex subset of $Gco\,(X) \times Gco\,(Y)$.

Next, we recall some notions and terminologies on the generalized Nash equilibrium for pure strategic games as in [2,3,7,13]. Let $I = \{1, 2, \ldots, n\}$ be a finite (or possibly countable) set of players. For each $i \in I$, let X_i be a nonempty set of actions. An *abstract economy* (or *generalized game*) $\mathcal{G} = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples (X_i, A_i, P_i) where X_i is a nonempty topological space (a choice set), $A_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a constraint correspondence and $P_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for $\mathcal{G}$ is a point $\hat{x} \in X = \Pi_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

As a proving tool for the main result of this paper, a multivalued generalization of the Németh fixed point theorem for geodesic convex sets in Hadamard manifolds is as follows:

Lemma 2.3. [4] Let X be a nonempty compact geodesic convex subset of a Hadamard manifold M, and $T : X \to 2^X$ be an upper semicontinuous multimap such that $T(x)$ is a nonempty closed geodesic convex subset of X for each $x \in X$. Then T has a fixed point $\bar{x} \in X$, that is, $\bar{x} \in T(\bar{x})$.

When $T : X \to 2^X$ is a single-valued function in Lemma 2.3, then T is clearly continuous and each singleton $T(x)$ is closed and geodesic convex. As a consequence, Lemma 2.3 reduces to the Németh fixed point theorem. Indeed, Németh [12] proved for single-valued version by using the Brouwer fixed point theorem instead of the Begle fixed point theorem in [4].

The following is well known in nonlinear analysis:

Lemma 2.4. [13] Let X and Y be two topological spaces, A an open subset of X, and $T_1, T_2 : X \to 2^Y$ be upper semicontinuous multimaps such that $T_2(x) \subseteq T_1(x)$ for all $x \in A$. Then a multimap $T : X \to 2^X$ defined by

$$T(x) := \begin{cases} T_1(x), & \text{if} \quad x \notin A; \\ T_2(x), & \text{if} \quad x \in A, \end{cases}$$

is also an upper semicontinuous multimap.

From now on, let M be a finite dimensional Hadamard manifold, and X be a nonempty geodesic convex subset of M. For the other standard notations and terminologies, we shall refer to Colao et al. [2], Kim [4,7], Kristály [8,9], Németh [12], and the references therein.

3. Equilibrium existence theorem for a generalized game

As an application of Lemma 2.3, we shall prove a new equilibrium existence theorem for a generalized game having geodesic convex values in a Hadamard manifold as follow:

Theorem 3.1. Let $\mathcal{G} = (X_i, A_i, P_i)_{i\in I}$ be a generalized game where I is a finite (possibly countable) set of agents such that for each $i \in I$,

(1) X_i is a nonempty compact geodesic convex subset of a Hadamard manifold M, and $X := \Pi_{i\in I} X_i$;

(2) $A_i : X \to 2^{X_i}$ be an upper semicontinuous multimap such that $A_i(x)$ is a nonempty closed and geodesic convex subset of X for each $x \in X$;

(3) $P_i : X \to 2^{X_i}$ be a multimap satisfies the irreflexivity, i.e., $x_i \notin P_i(x)$ for each $x \in X$;

(4) $A_i \cap P_i : X \to 2^{X_i}$ be an upper semicontinuous multimap such that $(A_i \cap P_i)(x)$ is a (possibly empty) closed and geodesic convex subset of X for each $x \in X$;

(5) the set $W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is (possibly empty) open.

Then $\mathcal{G}$ has an equilibrium choice $\hat{x} \in X$, that is, for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Proof. First, suppose that $W_i = \emptyset$ for all $i \in I$. We define a multimap $A : X \to 2^X$ by

$$A(x) := \Pi_{i\in I} A_i(x) \quad \text{for each } x \in X.$$

Then, by the assumption (2), A is an upper semicontinuous multimap such that $A(x)$ is a nonempty closed and geodesic convex subset of X for each $x \in X$. Therefore, by Lemma 2.3, there exists a fixed point $\hat{x} \in X$ for A, that is, for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ which completes the proof.

Next, suppose that I_o is a nonempty maximal subset of I such that W_i is nonempty for each $i \in I_o$. For each $i \in I_o$, we define a multimap $\phi_i : X \to 2^{X_i}$ by

$$\phi_i(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \in W_i; \\ A_i(x), & \text{if } x \notin W_i. \end{cases}$$

Then, for each $i \in I_o$, by the assumptions (2) and (4), we have $\phi_i(x)$ is a nonempty closed and geodesic convex subset of X_i for each $x \in X$.

Since $(A_i \cap P_i)(x) \subseteq A_i(x)$ for each $x \in X$, and W_i is open, by Lemma 2.4, ϕ_i is also upper semicontinuous.

Finally, we define a multimap $\Phi : X \to 2^X$ by

$$\Phi(x) := \Pi_{i\in I}\, \phi'_i(x) \quad \text{ for each } x \in X,$$

where $\phi'_i : X \to 2^{X_i}$ is defined by

$$\phi'_i(x) = \begin{cases} \phi_i(x), & \text{if } \; i \in I_o; \\ A_i(x), & \text{if } \; i \notin I_o. \end{cases}$$

Then, $\Phi : X \to 2^X$ is an upper semicontinuous such that $\Phi(x)$ is a nonempty closed and geodesic convex subset of X for each $x \in X$. Therefore, the multimap $\Phi : X \to 2^X$ satisfies the whole assumptions of Lemma 2.3 so that there exists a fixed point $\hat{x} \in X$ such that $\hat{x} \in \Phi(\hat{x})$, that is, $\hat{x}_i \in \phi'_i(\hat{x})$ for each $i \in I$.

Next, we shall check the two cases. Indeed, for each $i \in I_o$, $\hat{x}_i \in \phi'_i(\hat{x}) = \phi_i(\hat{x})$. If $\hat{x} \in W_i$, then we have

$$\hat{x}_i \in \phi_i(\hat{x}) = (A_i \cap P_i)(\hat{x}) \subseteq P_i(\hat{x})$$

which contradicts the irreflexivity assumption (3). Therefore, whenever $i \in I_o$, we should have $\hat{x} \notin W_i$ so that

$$\hat{x}_i \in \phi'_i(\hat{x}) = \phi_i(\hat{x}) = A_i(\hat{x}) \quad \text{and} \quad (A_i \cap P_i)(\hat{x}) = \emptyset.$$

In case of $i \notin I_o$, we have $W_i = \emptyset$ so that $(A_i \cap P_i)(\hat{x}) = \emptyset$, and $\hat{x}_i \in \phi'_i(\hat{x}) = A_i(\hat{x})$. Therefore, in either cases, we have that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$$

so that $\hat{x} \in X$ is an equilibrium choice for the game $\mathcal{G}$. This completes the proof. □

Remark 3.2. For each $i \in I$, when $P_i(x)$ is empty for each $x \in X$, then the assumptions (3)-(5) of Theorem 3.1 are automatically satisfied so that the equilibrium point $\hat{x} \in X$ is a fixed point for A_i for each $i \in I$. In this case, we can obtain Theorem 3.1 as a multivalued generalization of the Németh fixed point theorem for geodesic convex sets in Hadamard manifold.

It is well known that when a subset of n-dimensional Euclidean space with its usual flat metric is geodesic convex if and only if it is convex in the usual sense, and similarly for functions. Thus, by replacing the Fan-Glicksberg fixed point theorem instead of Lemma 2.3 in the proof of Theorem 3.1, we can obtain the following equilibrium existence theorem for a generalized game in locally convex Hausdorff topological vector space setting:

Theorem 3.3. Let $\mathcal{G} = (X_i, A_i, P_i)_{i\in I}$ be a generalized game where I is a finite (possibly countable) set of agents such that for each $i \in I$,

(1) X_i is a nonempty compact convex subset of a locally convex Hausdorff topological vector space M, and $X := \Pi_{i\in I} X_i$;
(2) $A_i : X \to 2^{X_i}$ be an upper semicontinuous multimap such that $A_i(x)$ is a nonempty closed and convex subset of X for each $x \in X$;
(3) $P_i : X \to 2^{X_i}$ be a multimap satisfies the irreflexivity, i.e., $x_i \notin P_i(x)$ for each $x \in X$;
(4) $A_i \cap P_i : X \to 2^{X_i}$ be an upper semicontinuous multimap such that $(A_i \cap P_i)(x)$ is a (possibly empty) closed convex subset of X for each $x \in X$;
(5) the set $W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is (possibly empty) open.

Then the game $\mathcal{G}$ has an equilibrium choice $\hat{x} \in X$, that is, for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Finally, we give an example of a non-convex 2-person game which is suitable for Theorem 3.1, but the previous equilibrium existence theorems in Ding-Kim-Tan [3] and Tan-Yuan [13] for compact games can not be applied:

Example 3.4. Let $\mathcal{G} = (X_i; A_i, P_i)_{i\in I}$ be a non-convex generalized game such that for each player $i \in \{1, 2\}$, the pure strategic space X_i is defined by

$$X_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1, x_2 \leq 1\};$$
$$X_2 := \{(\cos t, \sin t) \in \mathbb{R}^2 \mid 0 \leq t \leq \pi\}.$$

Then, X_1 is a compact (geodesic) convex subset of $\mathbb{R}^2$ in the usual sense, and X_2 is compact but not a convex subset of $\mathbb{R}^2$ in the usual sense. However, as remarked in [9], if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, then the set X_2 is geodesic convex with respect to the metric $g_{\mathbb{H}}$ being the image of a geodesic segment from $(\mathbb{H}^2, g_{\mathbb{H}})$.

For each player $i = 1, 2,$ the multimaps $A_i : X = X_1 \times X_2 \to 2^{X_i}$ and $P_i : X \to 2^{X_i}$ are defined as follows:

For each $\big((x_1, x_2), (y_1, y_2)\big) \in X$,

$$A_1\big((x_1, x_2), (y_1, y_2)\big) := \begin{cases} \big\{(\bar{x}_1, \bar{x}_2) \in X_1 \mid \bar{x}_1 \leq x_1\big\}, & \text{if} \quad x_1 \neq 0; \\ X_1, & \text{if} \quad x_1 = 0; \end{cases}$$

$$A_2\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{y}_1,\bar{y}_2) \in X_2 \mid \bar{y}_1 \le y_1\big\}, & \text{if} \quad y_1 \ne -1; \\ X_2, & \text{if} \quad y_1 = -1; \end{cases}$$

$$P_1\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{x}_1,\bar{x}_2) \in X_1 \mid \bar{x}_1 > x_1\big\}, & \text{if} \quad x_1 \ne 0; \\ \emptyset, & \text{if} \quad x_1 = 0; \end{cases}$$

$$P_2\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{y}_1,\bar{y}_2) \in X_2 \mid \bar{y}_1 > y_1\big\}, & \text{if} \quad y_1 \ne -1; \\ \emptyset, & \text{if} \quad y_1 = -1. \end{cases}$$

Then, it is clear that for each $i = 1,2$ and $\big((x_1,x_2),(y_1,y_2)\big) \in X$, $A_i\big((x_1,x_2),(y_1,y_2)\big)$ are nonempty closed and geodesic convex subsets of X_i, and $(x_1,x_2) \notin P_1\big((x_1,x_2),(y_1,y_2)\big)$ and $(y_1,y_2) \notin P_2\big((x_1,x_2),(y_1,y_2)\big)$. Therefore, the assumptions (1)-(3) of Theorem 3.1 are satisfied. In order to apply Theorem 3.1 to the game $\mathcal{G}$, it remains to show the assumptions (4) and (5) of Theorem 3.1. Indeed, the set $W_1 = \{x \in X \mid (A_1 \cap P_1)(x) \ne \emptyset\}$ is empty, and the set $W_2 = \{x \in X \mid (A_2 \cap P_2)(x) \ne \emptyset\}$ is also empty; thus the assumptions (4) and (5) of Theorem 3.1 is automatically satisfied. Therefore, all the assumptions of Theorem 3.1 for the generalized game $\mathcal{G}$ are satisfied so that we can obtain an equilibrium point $\big((0,1),(-1,0)\big) \in X = X_1 \times X_2$ for the generalized game $\mathcal{G} = (X_i; A_i, P_i)_{i \in I}$ such that for each $i = 1,2$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset; \quad \text{that is ,}$$

$$(0,1) \in A_1\big((0,1),(-1,0)\big) \quad \text{and} \quad (A_1 \cap P_1)\big((0,1),(-1,0)\big) = \emptyset;$$

$$(-1,0) \in A_2\big((0,1),(-1,0)\big) \quad \text{and} \quad (A_2 \cap P_2)\big((0,1),(-1,0)\big) = \emptyset.$$

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